

Linear Algebra and Calculus Exercises: Part II

CSCI 567 Machine Learning

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MULTIPLE-CHOICE QUESTIONS: one or more correct choices for each question.

Q1 Which of the following statements are true? PSD stands for positive semi-definite.

- (a) $XX^\top$ is a PSD matrix if and only if X is PSD.
- (b) If X and Y are PSD matrices, then so is $\lambda X + \mu Y$ for any $\lambda, \mu \in \mathbb{R}$.
- (c) If $X - Y$ and $X + Y$ are PSD matrices, then so are X and Y .
- (d) All eigenvalues of a symmetric PSD matrix are non-negative.

Q2 Suppose A and B are two positive definite matrices. Which matrix may NOT be positive definite?

- (a) A^{-1}
- (b) $A + B$
- (c) $AA^\top$
- (d) $A - B$

SHORT-ANSWER QUESTION. *The following questions use linear algebra and calculus in ML formulations. They particularly test your knowledge of gradients of multivariate functions.*

Q3 Consider the following optimization problem:

$$\mathbf{w}_* = \arg \min_{\mathbf{w} \in \mathbb{R}^d} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|_2^2 + \mathbf{w}^T \mathbf{M} \mathbf{w}$$

Here, $\mathbf{X} \in \mathbb{R}^{n \times d}$, $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{M} \in \mathbb{R}^{d \times d}$ is a positive definite matrix and $\|\cdot\|_2$ stands for the ℓ_2 norm. Find the closed form solution for $\mathbf{w}_*$. Proceed in a similar way as how we derived the general least-squares solution in class. (This optimization problem is a generalization of ℓ_2 regularization, which we will see in class.)

Q3 Assume we have a training set $(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n) \in \mathbb{R}^d \times \mathbb{R}$, where each outcome y_i is generated by a probabilistic model $\mathbf{w}_*^T \mathbf{x}_i + \epsilon_i$ with ϵ_i being an independent Gaussian noise with zero-mean and variance σ^2 for some $\sigma > 0$. In other words, the probability of seeing any outcome $y \in \mathbb{R}$ given $\mathbf{x}_i \in \mathbb{R}^d$ is

$$\Pr(y \mid \mathbf{x}_i; \mathbf{w}_*, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(y - \mathbf{w}_*^T \mathbf{x}_i)^2}{2\sigma^2}\right).$$

Assume σ is fixed and given, find the maximum likelihood estimation for $\mathbf{w}_*$. In other words, first write down the probability of seeing the outcomes $y_1, \dots, y_n$ given $\mathbf{x}_1, \dots, \mathbf{x}_n$ as a function of the value of $\mathbf{w}_*$; then find the value of $\mathbf{w}_*$ that maximizes this probability. You can assume $\mathbf{X}^T \mathbf{X}$ is invertible, where $\mathbf{X}$ is the data matrix with each row corresponding to the features of an example. You may find it helpful to review the steps we took in Lecture 2 to find the maximum likelihood solution for the logistic model.