CSCI 567 Discussion
Linear Algebra and Numpy Review I

(Slides adapted from Nandita Bhaskhar's slides for CS229 at Stanford)

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Outline

- Basic Concepts and Notation
- Matrix Multiplications
- Operations and Properties
- Matrix Calculus

Basic Concepts and Notation

Basic Notation

- By $x \in \mathbb{R}^n$, we denote a vector with n entries.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- By $A \in \mathbb{R}^{m \times n}$, we denote a matrix with m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a^1 & a^2 & \cdots & a^n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} --- & a_1^T & --- \\ --- & a_2^T & --- \\ & \vdots & \\ --- & a_m^T & --- \end{bmatrix}$$

Special Matrices

Identity matrix

$$I_n \in \mathbb{R}^{n \times n}$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

For all $A \in \mathbb{R}^{m \times n}$, $AI_n = A = I_m A$.

Diagonal matrix

$$D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$$

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n \end{bmatrix}$$

Clearly, $I = \mathrm{diag}(1,1,\ldots,1)$.

Matrix Multiplication

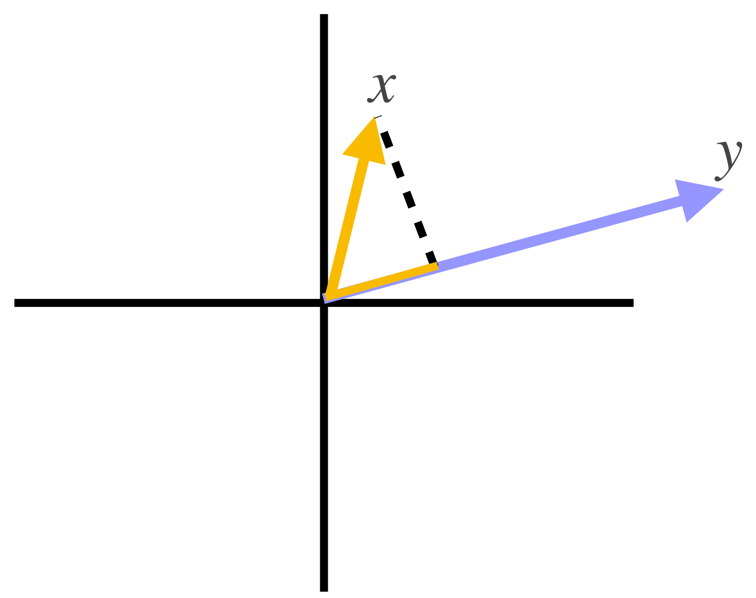
Vector-Vector Product

Inner Product or Dot Product

$$x^T y \in \mathbb{R} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i \,.$$

Intuition

$x^T y =$ (Length of projected x)·(Length of y)



Vector-Vector Product

Outer Product

$$xy^T \in \mathbb{R}^{m\times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

$$\begin{bmatrix} & x_1 & & x_1 & \cdots & & x_1 \\ y_1 & \vdots & y_2 & \vdots & \ddots & y_n & \vdots \\ & x_m & & x_m & \cdots & & x_m \end{bmatrix}$$

$$\begin{bmatrix} x_1 & (\cdots & y^T & \cdots) \\ x_2 & (\cdots & y^T & \cdots) \\ \vdots & \vdots & \vdots & \vdots \\ x_m & (\cdots & y^T & \cdots) \end{bmatrix}$$

Matrix-Vector Product

View 1: Write A by rows

$$y = Ax = \begin{bmatrix} \text{---} & a_1^T & \text{---} \\ \text{---} & a_2^T & \text{---} \\ & \vdots & \\ \text{---} & a_m^T & \text{---} \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}.$$

Set of inner products with each row vector

Matrix-Vector Product

View 2: Write A by columns

$$y = Ax = \begin{bmatrix} | & | & & | \\ a^1 & a^2 & \cdots & a^n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | \\ a^1 \\ | \end{bmatrix} x_1 + \begin{bmatrix} | \\ a^2 \\ | \end{bmatrix} x_2 + \ldots + \begin{bmatrix} | \\ a^n \\ | \end{bmatrix} x_n .$$

Linear combination of column vectors

Vector-Matrix Product

View 1: Write A by columns

$$y^T = x^T A = x^T \begin{bmatrix} | & | & & | \\ a^1 & a^2 & \cdots & a^n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} x^T a^1 & x^T a^2 & \cdots & x^T a^n \end{bmatrix}$$

Set of inner products with each column vector

Vector-Matrix Product

View 2: Write A by rows

$$y^T = x^T A = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} \text{---} & a_1^T & \text{---} \\ \text{---} & a_2^T & \text{---} \\ & \vdots & \\ \text{---} & a_m^T & \text{---} \end{bmatrix}$$

$$= x_1 \begin{bmatrix} \text{---} & a_1^T & \text{---} \end{bmatrix} + x_2 \begin{bmatrix} \text{---} & a_2^T & \text{---} \end{bmatrix} + \ldots + x_m \begin{bmatrix} \text{---} & a_m^T & \text{---} \end{bmatrix}$$

Linear combination of row vectors

Matrix-Matrix Multiplication

View 1: Set of inner products

$$C = AB = \begin{bmatrix} \text{---} & a_1^T & \text{---} \\ \text{---} & a_2^T & \text{---} \\ & \vdots & \\ \text{---} & a_m^T & \text{---} \end{bmatrix} \begin{bmatrix} | & | & & | \\ b^1 & b^2 & \cdots & b^n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b^1 & a_1^T b^2 & \cdots & a_1^T b^n \\ a_2^T b^1 & a_2^T b^2 & \cdots & a_2^T b^n \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b^1 & a_m^T b^2 & \cdots & a_m^T b^n \end{bmatrix}$$

Matrix-Matrix Multiplication

View 2: Sum of outer products

$$C = AB = \begin{bmatrix} | & | & & | \\ a^1 & a^2 & \cdots & a^n \\ | & | & & | \end{bmatrix} \begin{bmatrix} --- & b_1^T & --- \\ --- & b_2^T & --- \\ & \vdots & \\ --- & b_n^T & --- \end{bmatrix} = a^1 b_1^T + a^2 b_2^T + \cdots a^n b_n^T = \sum_{i=1}^{n} a^i b_i^T$$

Matrix-Matrix Multiplication

View 3: Set of matrix-vector products

$$C = AB = \boxed{A}\begin{bmatrix} | & | & & | \\ b^1 & b^2 & \cdots & b^n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ab^1 & Ab^2 & \cdots & Ab^n \\ | & | & & | \end{bmatrix}$$

Matrix-Matrix Multiplication

View 4: Set of vector-matrix products

$$C = AB = \begin{bmatrix} \text{---} & a_1^T & \text{---} \\ \text{---} & a_2^T & \text{---} \\ & \vdots & \\ \text{---} & a_m^T & \text{---} \end{bmatrix} B = \begin{bmatrix} \text{---} & a_1^T B & \text{---} \\ \text{---} & a_2^T B & \text{---} \\ & \vdots & \\ \text{---} & a_m^T B & \text{---} \end{bmatrix}$$

Matrix-Matrix Multiplication

Properties

- Associative: $(AB)C = A(BC)$.
- Distributive: $A(B + C) = AB + AC$.
- In general, **not** commutative; it can be the case that $AB \neq BA$.

Exercise

- Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are all D-dimensional vectors, and $X \in \mathbb{R}^{N \times D}$ is a matrix where the n-th row is $\mathbf{x}_n^\top$. Then which of the following identities are correct?

A. $X^\top X = \sum_{n=1}^{N} \mathbf{x}_n \mathbf{x}_n^\top$

B. $X^\top X = \sum_{n=1}^{N} \mathbf{x}_n^\top \mathbf{x}_n$

C. $XX^\top = \sum_{n=1}^{N} \mathbf{x}_n \mathbf{x}_n^\top$

D. $XX^\top = \sum_{n=1}^{N} \mathbf{x}_n^\top \mathbf{x}_n$

$$\underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_N \\ | & & | \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} - & x_1 & - \\ & \vdots & \\ - & x_N & - \end{bmatrix}}_{X}$$

mat. mul. view 2 $\Rightarrow X^T X = \sum_{n=1}^{N} x_n x_n^T$

mat. mul. view 1 $\Rightarrow XX^T = \begin{bmatrix} x_1^T x_1 & \cdots & x_1^T x_N \\ \vdots & & \\ x_N^T x_1 & \cdots & x_N^T x_N \end{bmatrix}$

Operations and Properties

Transpose

The transpose of a matrix results from 'flipping' the rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- Properties:
  - $(A^T)^T = A$.
  - $(AB)^T = B^T A^T$.
  - $(A + B)^T = A^T + B^T$
- If $A = A^T$, then A is a symmetric matrix
- If $A = -A^T$, then A is an anti-symmetric matrix

Trace

The trace of a square matrix is the sum of its diagonal elements

$$\mathrm{tr}A = \sum_{i=1}^{n} A_{ii}\,.$$

- Properties ($A, B, C \in \mathbb{R}^{n\times n}$):
  - $\mathrm{tr}A = \mathrm{tr}A^T$.
  - $\mathrm{tr}(A+B) = \mathrm{tr}A + \mathrm{tr}B$.
  - $\mathrm{tr}(tA) = t\ \mathrm{tr}A$
  - $\mathrm{tr}AB = \mathrm{tr}BA$
  - $\mathrm{tr}ABC = \mathrm{tr}BCA = \mathrm{tr}CAB$, and so on.

Norms

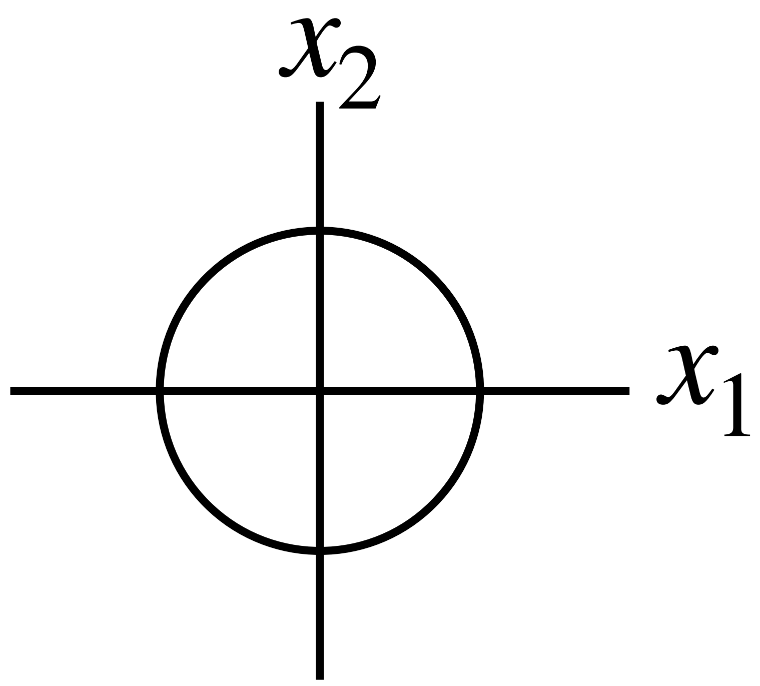
- Informally, norm of a vector measures the 'length' of the vector.

- Formally, any function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 4 properties for $x, y \in \mathbb{R}^n$:
  - Non-negativity: $f(x) \geq 0$
  - Definiteness: $f(x) = 0$ iff $x = 0$
  - Homogeneity: $f(tx) = |t| f(x)$
  - Triangle inequality: $f(x + y) \leq f(x) + f(y)$

Examples of Norms

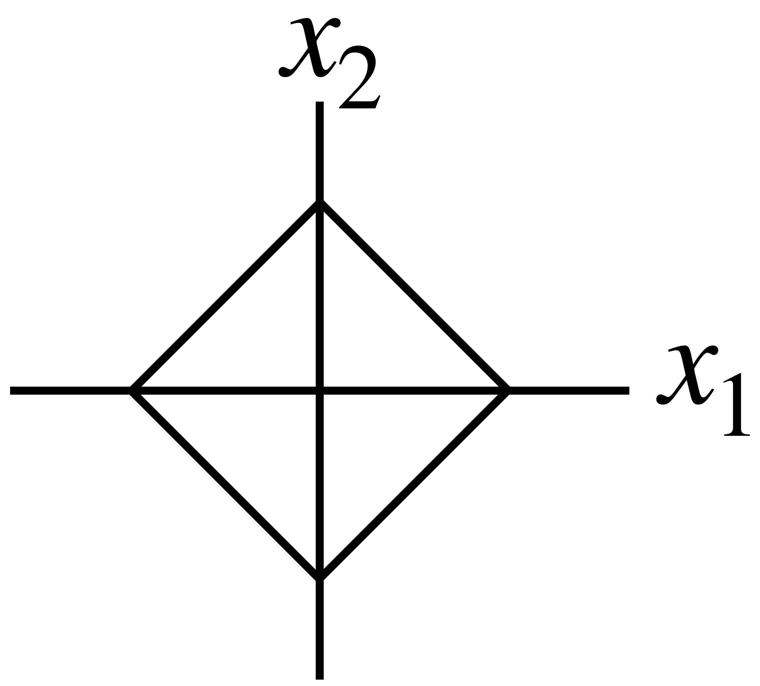
- Euclidean or ℓ_2–norm:

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x}$$



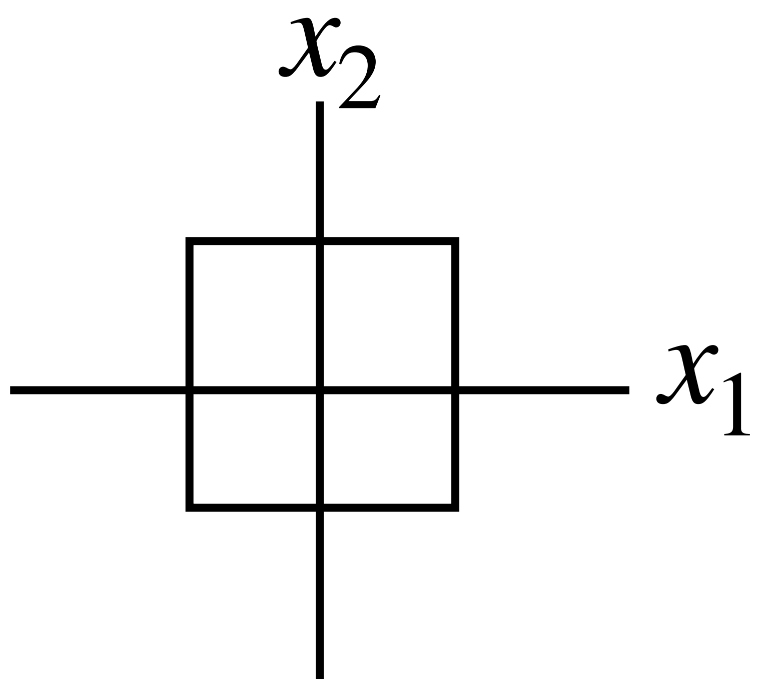
- ℓ_1–norm:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$



- ℓ_∞–norm:

$$\|x\|_\infty = \max_i |x_i|$$

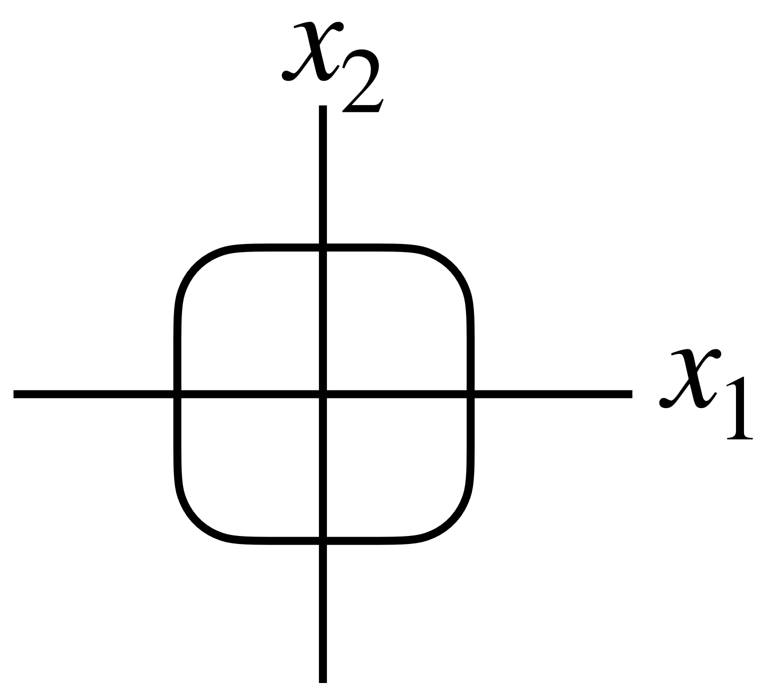


ℓ_p−Norms

- Family of ℓ_p−norms, parameterized by a real number $p \geq 1$:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

- For $p \geq 2$:



Matrix Norms

- Frobenius norm:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}$$

$$= \sqrt{\sum_{i=1}^{m} \|a_i\|_2^2} = \sqrt{\sum_{j=1}^{n} \|a^j\|_2^2}$$

$$= \sqrt{\mathrm{tr}(A^T A)}$$

Linear Combinations and Span

- The span of a set of vectors $\{x_1, x_2, \ldots x_n\}$ is the set of all vectors that can be expressed as a linear combination of $\{x_1, \ldots, x_n\}$. That is,

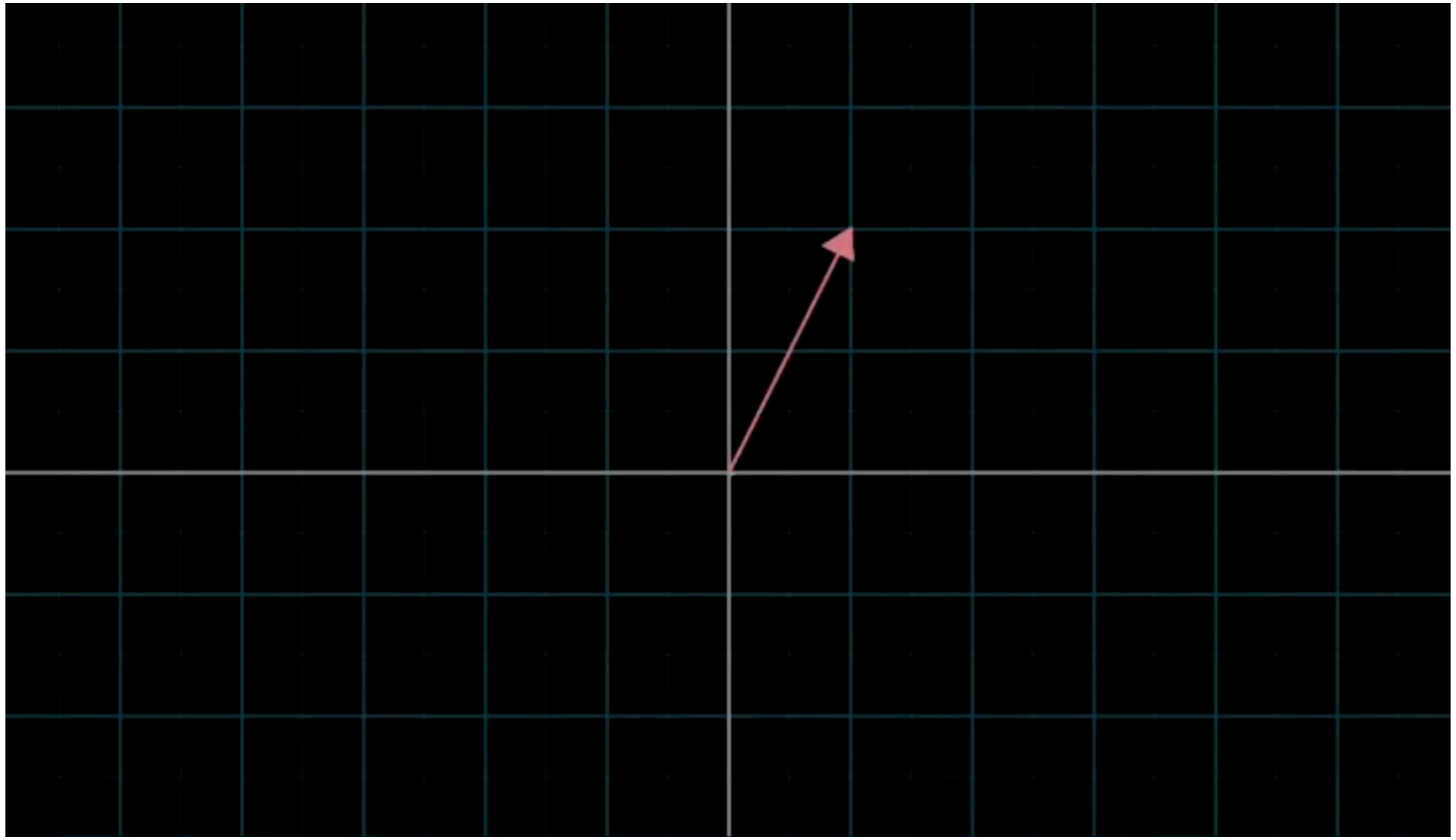
$$\mathrm{span}(\{x_1, \ldots x_n\}) = \left\{ v : v = \sum_{i=1}^{n} \alpha_i x_i, \;\; \alpha_i \in \mathbb{R} \right\}$$

- The span of column vectors of a matrix is known as the column space.

- Similarly, the span of row vectors is known as the row space.

Linear Combinations and Span

Visualization



Source: 3Blue1Brown via YouTube: https://tinyurl.com/2p9e3waa

Linear Independence

- A set of vectors $\{x_1, x_2, \ldots x_n\} \subset \mathbb{R}^m$ is said to be (linearly) dependent if one vector belonging to the set can be represented as a linear combination of the remaining vectors; that is, if

$$x_n = \sum_{i=1}^{n-1} \alpha_i x_i$$

for some scalar values $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$.

- Otherwise, the vectors are (linearly) independent.

Rank

- Column rank: largest number of columns that constitute a linearly independent set.
- Row rank: largest number of rows that constitute a linearly independent set.
- Column rank of any matrix is equal to its row rank.
- Both quantities collectively referred to as the rank of the matrix.

- Properties ($A \in \mathbb{R}^{m \times n}$):
  - $\mathrm{rank}(A) \leq \min(m, n)$. If $\mathrm{rank}(A) = \min(m, n)$, A is said to be full rank.
  - $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.
  - For $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$, $\mathrm{rank}(AB) \leq \min(\mathrm{rank}(A), \mathrm{rank}(B))$.
  - For $A, B \in \mathbb{R}^{m \times n}$, $\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$.

Inverse of a Square Matrix

- The inverse of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted A^{-1}, is the unique matrix such that $A^{-1}A = I_n = AA^{-1}$.
- A must be full rank for its inverse to exist.
- A is invertible or non-singular if A^{-1} exists and non-invertible or singular otherwise.

- Properties ($A, B \in \mathbb{R}^{n \times n}$ are non-singular):
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(A^{-1})^T = (A^T)^{-1}$, denoted by A^{-T}

Determinant

Intuition

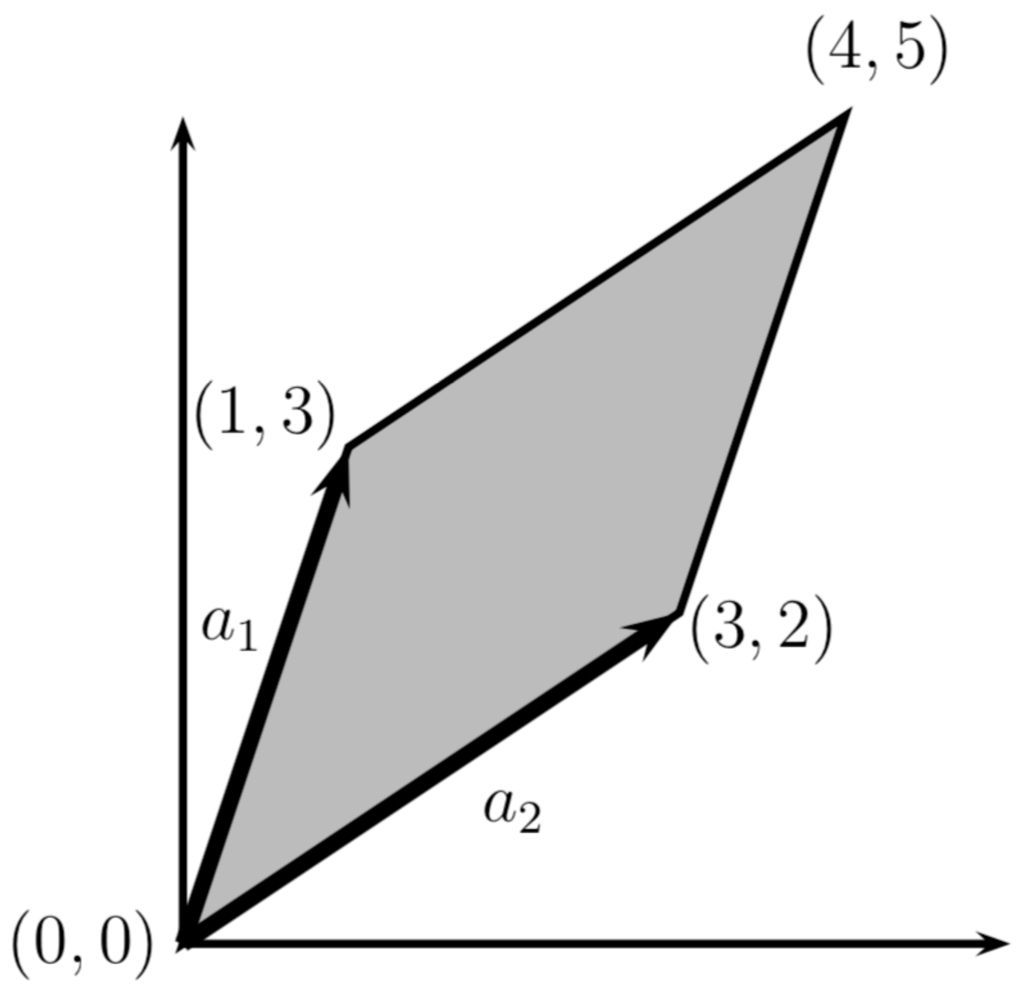
- Let $A \in \mathbb{R}^{n \times n}$, a_i^T denotes its ith row; consider the set of points $S \subset \mathbb{R}^n$:

$$S = \{v \in \mathbb{R}^n : v = \sum_{i=1}^{n} \alpha_i a_i \ \ (0 \leq \alpha_i \leq 1; \quad i = 1, \ldots, n)\}$$

- The absolute value of the determinant of A gives the 'volume' of the set S

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad a_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



Determinant

(Recursive) Formula

- Let $A \in \mathbb{R}^{n\times n}$, $A_{\backslash i,\backslash j} \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix that results from deleting the ith row and jth column from A

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{\backslash i,\backslash j}| \qquad (\forall\ j \in 1,\ldots,n)$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{\backslash i,\backslash j}| \qquad (\forall\ i \in 1,\ldots,n)$$

- Equations for small matrices:

$$\Big|[a_{11}]\Big| = a_{11}$$

$$\left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right| = a_{11}a_{22} - a_{12}a_{21}$$

Determinant

Properties

- Properties ($A, B \in \mathbb{R}^{n \times n}$):
  - $|A| = |A^T|$
  - $|AB| = |A||B|$
  - $|A| = 0$ iff A is singular
  - For non-singular A, $|A^{-1}| = 1/|A|$

Exercise

- Which identities are **NOT** correct for real-valued matrices A, B, and C? Assume that inverses exist and multiplications are legal.

$(I-A)(I+A) \neq I$

$(AB)^T = B^T A^T$

A. $(AB)^{-1} = B^{-1}A^{-1}$

B. $(I+A)^{-1} = I - A$

C. $\text{tr}(AB) = \text{tr}(BA)$

D. $(AB)^\top = A^\top B^\top$

Exercise

- Consider some vector $x \in \mathbb{R}^n$. What is the rank of the matrix xx^T?

$$xx^T = \begin{bmatrix} | & & | \\ x_1 x & \cdots & x_n x \\ | & & | \end{bmatrix}$$

$\Rightarrow$ Rank $= 1$

Matrix Calculus

Gradient

- Suppose $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is a scalar function that takes as input a matrix $A \in \mathbb{R}^{m \times n}$

- The gradient of f with respect to A is the $(m \times n)$ matrix of partial derivatives:

$$\nabla_A f(A) = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \frac{\partial f(A)}{\partial A_{12}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \frac{\partial f(A)}{\partial A_{21}} & \frac{\partial f(A)}{\partial A_{22}} & \cdots & \frac{\partial f(A)}{\partial A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial A_{m1}} & \frac{\partial f(A)}{\partial A_{m2}} & \cdots & \frac{\partial f(A)}{\partial A_{mn}} \end{bmatrix}$$

Gradient

- If the input is just a vector $x \in \mathbb{R}^n$,

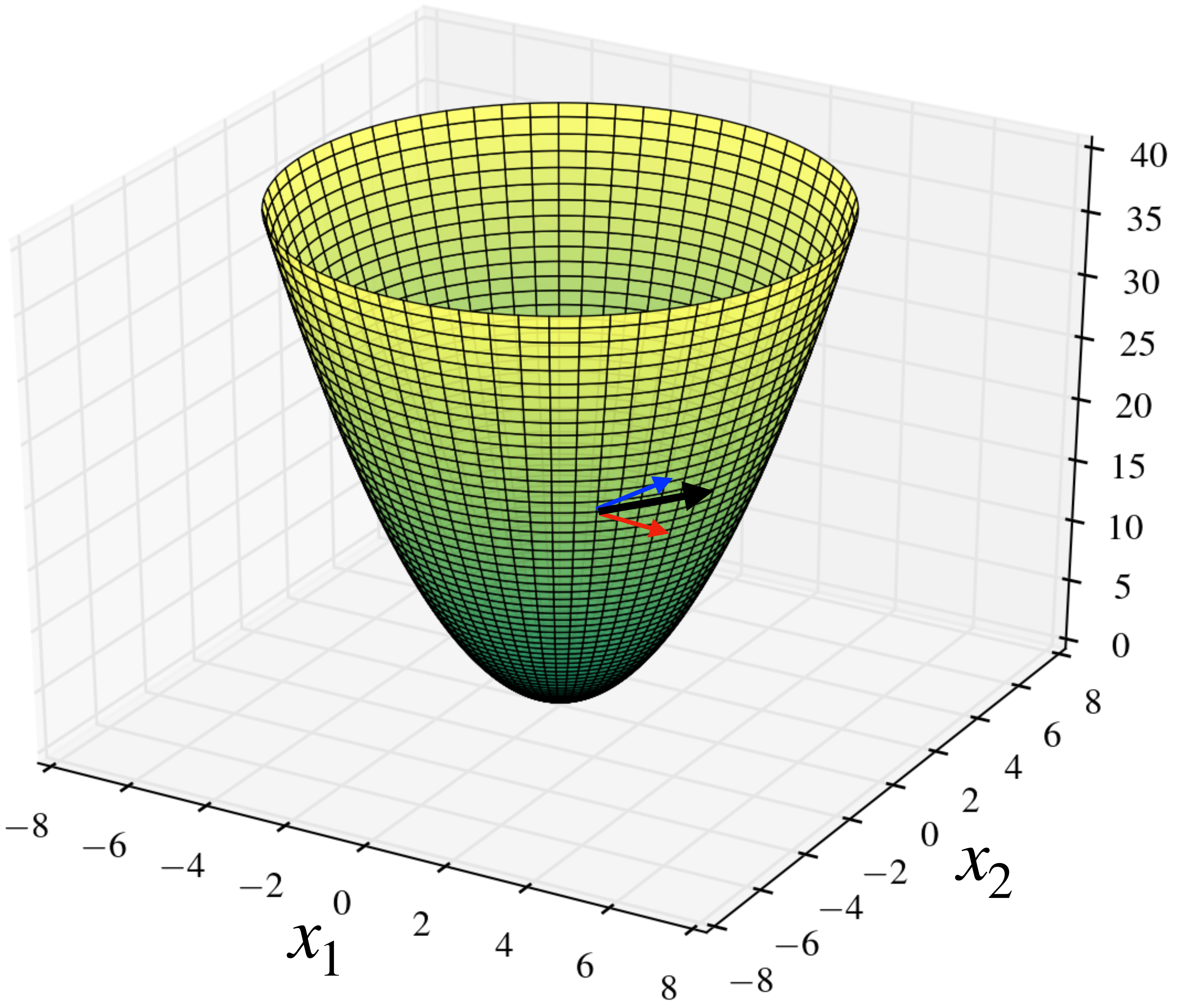
$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

- Properties of partial derivatives extend here:
  - $\nabla_x (f(x) + g(x)) = \nabla_x f(x) + \nabla_x g(x)$.
  - For $t \in \mathbb{R}$, $\nabla_x (t\, f(x)) = t \nabla_x f(x)$.

Gradient

Visual Example

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{bmatrix}$$



Hessian

- Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar function that takes as input a vector $x \in \mathbb{R}^n$
- The Hessian of f with respect to x is the $(n \times n)$ matrix of partial derivatives:

$$\nabla_x^2 f(x) \in \mathbb{R}^{n\times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

- It is symmetric (provided the second partial derivatives are continuous).

Jacobian

- Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a vector function that takes as input a vector $x \in \mathbb{R}^n$
- The Jacobian of f with respect to x is the $(m \times n)$ matrix of partial derivatives:

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} & \frac{\partial f(x)}{\partial x_2} & \cdots \frac{\partial f(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla_x^T f_1(x) \\ \nabla_x^T f_2(x) \\ \vdots \\ \nabla_x^T f_m(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix}$$

Gradient of a Linear Function

- For $x \in \mathbb{R}^n$, let $f(x) = b^T x\ (= x^T b)$ for some known vector $b \in \mathbb{R}^n$. Then,

$$f(x) = \sum_{i=1}^{n} b_i x_i$$

- This gives:

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^{n} b_i x_i = b_k$$

$$\nabla_x b^T x = b$$

- Analogous to single variable calculus, where $\dfrac{\partial\,(ax)}{\partial x} = a$

Jacobian of a Linear Function

- For $x \in \mathbb{R}^n$, let $f(x) = Ax$ for some known matrix $A \in \mathbb{R}^{m \times n}$. Then,

$$f_i(x) = a_i^T x \quad \forall i = 1, \cdots, m$$

- This gives:

$$\nabla_x f_i(x) = a_i$$

$$\nabla_x f(x) = \begin{bmatrix} \text{---} & a_1^T & \text{---} \\ \text{---} & a_2^T & \text{---} \\ & \vdots & \\ \text{---} & a_m^T & \text{---} \end{bmatrix} = A$$

Gradient of a Quadratic Function

- For $x \in \mathbb{R}^n$, let $f(x) = x^T A x$ for some known matrix $A \in \mathbb{R}^{n \times n}$. Then,

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

- Using previous slides, product rule for $f(x) = g(x)^T x$, with $g(x) = A^T x$, we get:

$$\begin{aligned} \nabla_x f(x) &= \nabla_x^T g(x) x + \nabla_x^T x g(x) \\ &= (A^T)^T x + I^T A^T x \\ &= (A + A^T) x \end{aligned}$$

- This gives the Hessian:

$$\nabla_x^2 f(x) = A + A^T$$

Exercise

- A function $f : \mathbb{R}^{n\times 1} \rightarrow \mathbb{R}$ is defined as $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x}$ for some $\mathbf{b} \in \mathbb{R}^{n\times 1}$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$. What is the derivative $\dfrac{\partial f}{\partial \mathbf{x}}$ (also called the gradient $\nabla f(\mathbf{x})$)?

Ans: $(A+A^T)x + b$

Exercise

- A function $f : \mathbb{R}^{n\times 1} \rightarrow \mathbb{R}$ is defined as $f(\mathbf{A}) = \mathbf{x}^\top \mathbf{A}\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^{n\times 1}$. What is the derivative $\dfrac{\partial f}{\partial \mathbf{A}}$?

$$f(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial f}{\partial A_{ij}} = x_i x_j$$

$$\frac{\partial f}{\partial A} = x x^T$$

Questions?

Next Week: Probability Review