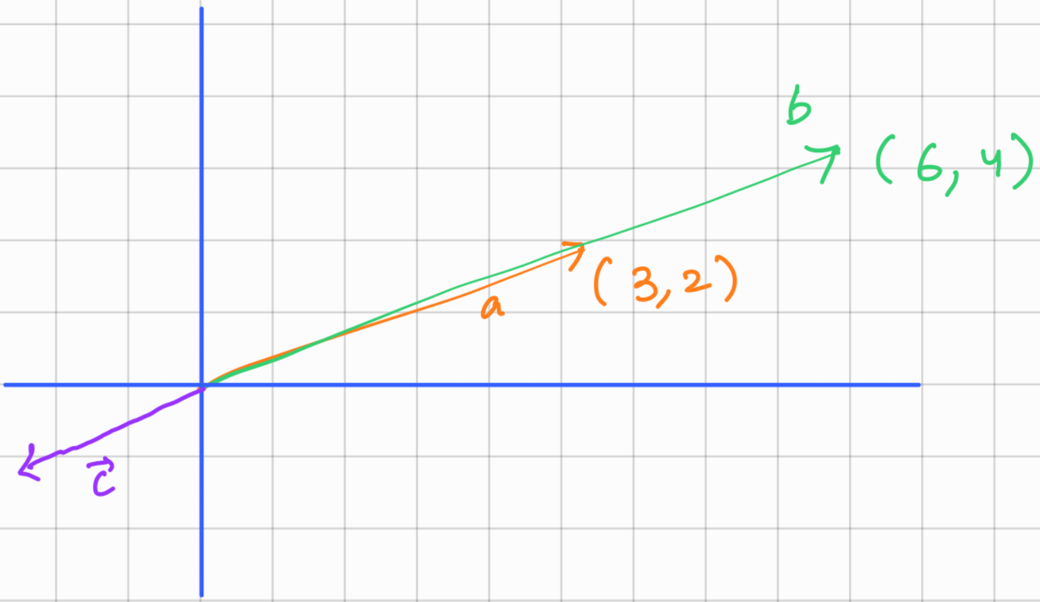
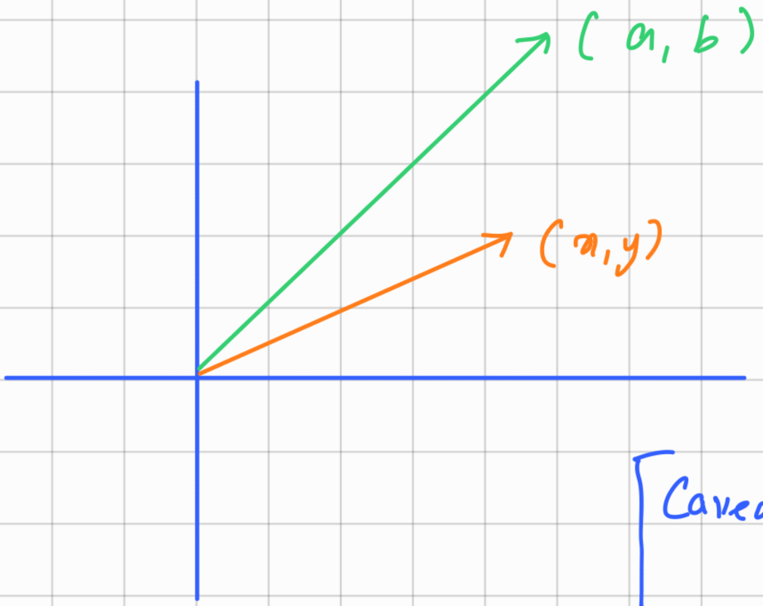
ALIGNED VECTORS



$$\vec{b} = 2\vec{a}$$

or $\vec{a} = \frac{1}{2}\vec{b}$

$$\vec{c} = -\frac{1}{2}\vec{a} = -\frac{1}{4}\vec{b}$$



Check?

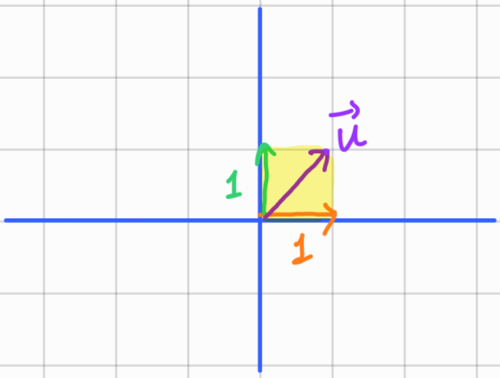
$$\frac{x}{a} = \frac{y}{b} \quad \text{or} \quad \frac{x}{y} = \frac{a}{b}.$$

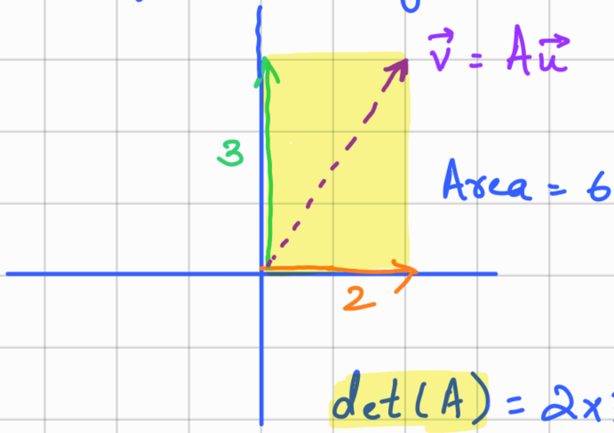
[Caveat: $x^2 + a^2 \neq 0$
$y^2 + b^2 \neq 0$]

RECAP : GEOMETRY OF SPECIAL MATRICES

STRETCH:

$$\overset{A}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \overset{\vec{u}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{\vec{v}}{\begin{bmatrix} 2x \\ 3y \end{bmatrix}}$$





$\det(A) = 2 \times 3 = 6$

Any vector $\vec{u}$ such that $\vec{v} = A\vec{u}$ and $\vec{u}$ align?

NO ALGEBRA PLEASE !

Check? $\quad \dfrac{2x}{x} = \dfrac{3y}{y}$?

$$\Rightarrow 2 = 3 \; ?$$

Caveat: $\quad 2x^2 + x^2 \neq 0$ or $\vec{u} = \begin{bmatrix} 0 \\ y \end{bmatrix}$

$$\vec{v} = A\vec{u} = \begin{bmatrix} 0 \\ 3y \end{bmatrix} = 3\vec{u}$$

$3y^2 + y^2 \neq$ or $\vec{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$$\vec{v} = A\vec{u} = \begin{bmatrix} 2x \\ 0 \end{bmatrix} = 2\vec{u}$$

$\vec{v} = \lambda \vec{u}$ or $A\vec{u} = \lambda \vec{u} = \lambda I \vec{u}$

$$\Rightarrow (A - \lambda I)\vec{u} = \vec{0}$$

$\Rightarrow (A - \lambda I)$ is not invertible.

[why? Is $(A-\lambda I)^{-1}\vec{0} = \vec{u}$? or $2\vec{u}$ or $k\vec{u}$?]

So, $\det(A - \lambda I) = \vec{0}$

$$\det \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \underbrace{(2-\lambda)(3-\lambda)}_{} = 0$$

Characteristic Polynomial of matrix A.

$\Rightarrow \lambda = 2$ or 3.

How to recover the directions?

$(A - 2I)\vec{u} = \vec{0} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ x = anything, $y = 0$ $\rightarrow$

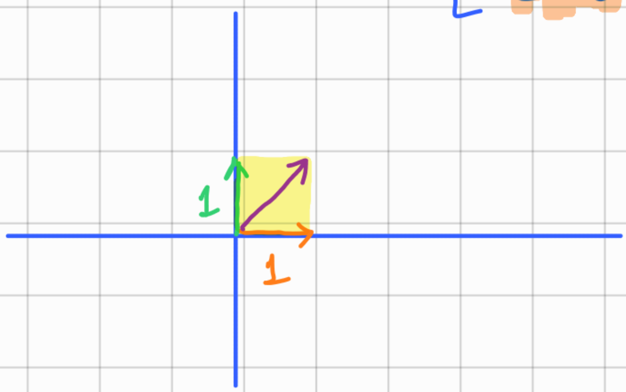
$(A - 3I)\vec{u} = 0 \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $x = 0$, y = anything $\uparrow$

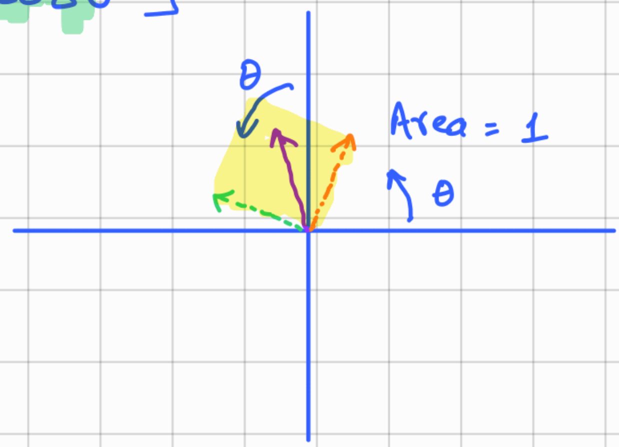
ROTATION
REFECTION

$\Big\}$ ORTHOGONAL / ORTHONORMAL / UNITARY

(ROTATION) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Verify
$A^T A = A A^T = I$





$\det(A) = \cos^2\theta + \sin^2\theta = 1$

Any vector that $\vec{v} = A\vec{u}$ that has same direction as $\vec{u}$?

Note: $\vec{v}$ is $\vec{u}$ rotated by θ degrees CCW/ACW

$$\vec{v} = \lambda\vec{u} \Rightarrow A\vec{u} = \lambda\vec{u} = \lambda I\vec{u} \Rightarrow (A - \lambda I)\vec{u} = \vec{0}$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\Rightarrow (\lambda - \cos\theta)^2 + \sin^2\theta = 0 \Rightarrow \lambda = \cos\theta \pm i\sin\theta.$$

So λ is not real, i.e. we have complex e. values.

For what value of θ is λ real?

$\theta = \frac{\pi}{2}$? $\quad$ $\theta = (2n+1)\pi$? $\quad$ $\theta = 2n\pi$?

$\lambda = -1$ $\quad$ $\lambda = 1$

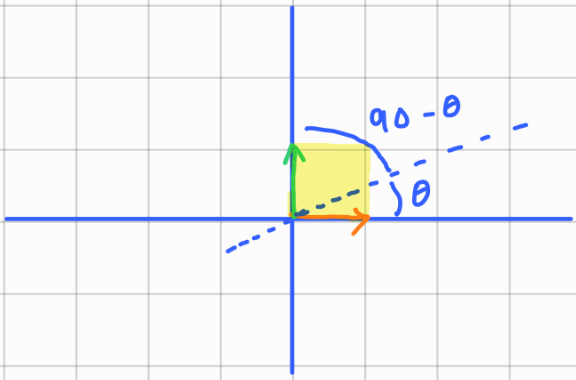
$A = -I$ $\quad$ $A = I$

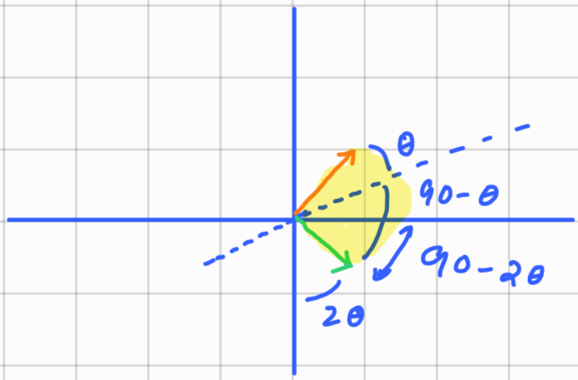
$$B = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

(REFLECTION)

Verify

$$B^T B = B B^T = I$$





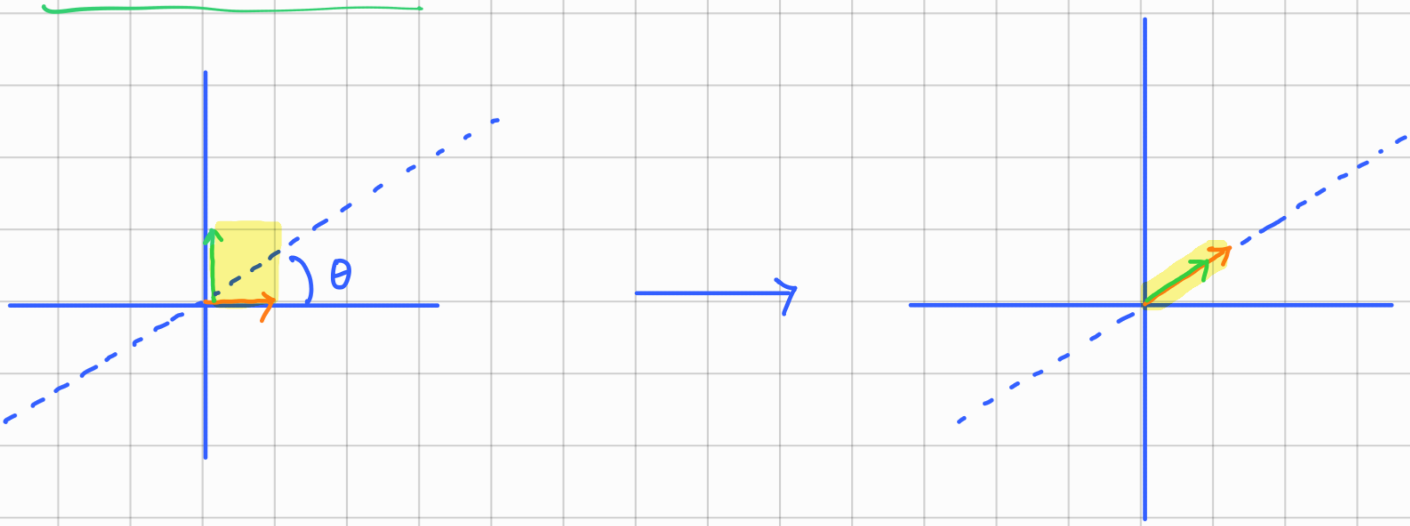
$$\det(B) = -\cos^2 2\theta - \sin^2 2\theta = -1$$

$$B\vec{u} = \lambda \vec{u} \Rightarrow (B - \lambda I)\vec{u} = 0$$

$$\Rightarrow \det(B - \lambda I) = 0$$

$$\Rightarrow \begin{bmatrix} \cos 2\theta - \lambda & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta - \lambda \end{bmatrix}$$

PROJECTION :-



$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$\det(P) = \cos^2\theta\sin^2\theta - (\cos\theta\sin\theta)^2$$

$$= \cos^2\theta\sin^2\theta - \cos^2\theta\sin^2\theta = 0$$

Verify $P = P^T$ & $P^2 = P$

Eigenvectors and eigenvalues

$$\det\left(\begin{bmatrix} \cos^2\theta - \lambda & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta - \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (\cos^2\theta - \lambda)(\sin^2\theta - \lambda) = \cos^2\theta\sin^2\theta$$

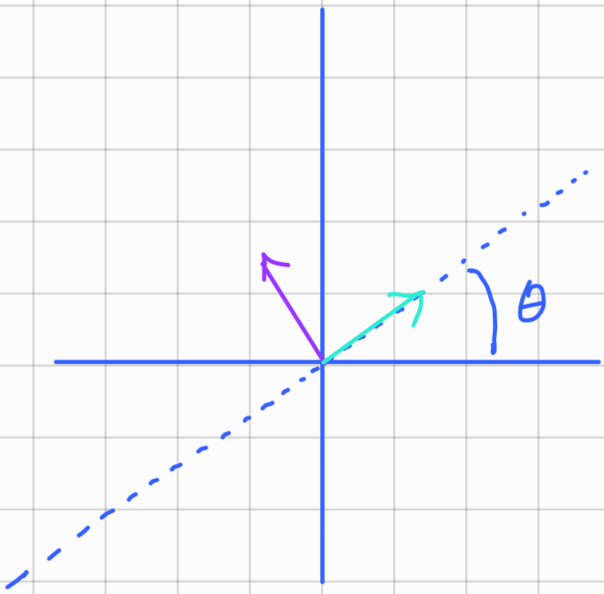
$$\Rightarrow \lambda^2 - \lambda + \cancel{\cos^2\theta\sin^2\theta} = \cancel{\cos^2\theta\sin^2\theta}$$

$$\Rightarrow \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0, 1$$

For $\lambda = 0$,

$$\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -s \\ +c \end{bmatrix}$$



For $\lambda = 1$

$$\begin{bmatrix} c^2-1 & cs \\ cs & s^2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ s \end{bmatrix}$$

Verify : $\begin{bmatrix} c^2-1 & cs \\ cs & s^2-1 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c^3 - c + cs^2 \\ c^2 s + s^3 - s \end{bmatrix}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ||$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c - c \\ s - s \end{bmatrix} = \begin{bmatrix} c(c^2+s^2) - c \\ s(c^2+s^2) - s \end{bmatrix}$$

(PSD)

POSITIVE SEMI-DEFINITE MATRICES

Symmetric matrices A with the properties :-

i) $x^T A x \geq 0$

ii) $\text{eigs}(A) \geq 0$

iii) $A = BB^T = \sum_i b_i b_i^T$ for some matrix B.

$\left.\right\} A \succeq 0$

iv) $A = U \Sigma U^T$

where $UU^T = U^T U = \mathbb{I}$

Σ is a diagonal matrix

$\Sigma_{ii} \geq 0$

Few Observations

(a) $S = M^T M$ is always PSD.

$$x^T S x = x^T M^T M x = (Mx)^T (Mx)$$
$$= \|Mx\|^2 \geq 0$$

(b) Projection matrix is always PSD

why? eigenvalues are $+1$ & 0

(c) $A, B \succeq 0 \Rightarrow \alpha A + \beta B \succeq 0$ if $\alpha, \beta \geq 0$

$$x^T(\alpha A + \beta B)x = (\alpha x^T A + \beta x^T B)x$$

$$= \underbrace{\alpha x^T A x}_{\geq 0} + \underbrace{\beta x^T B x}_{\geq 0} \geq 0$$

(d) $C = \left[\begin{array}{c|c} A_{m\times m} & O_{m\times n} \\ \hline O_{n\times m} & B_{n\times n} \end{array}\right] \succeq 0 \Leftarrow A, B \succeq 0$

prove it !

Let $\vec{x} = \left[\begin{array}{c} \vec{x}_1 \\ \hline \vec{x}_2 \end{array}\right]$

$$x^T C x = [\vec{x}_1 \,\vdots\, \vec{x}_2] \left[\begin{array}{c|c} A_{m\times m} & O_{m\times n} \\ \hline O_{n\times m} & B_{n\times n} \end{array}\right] \left[\begin{array}{c} \vec{x}_1 \\ \hline \vec{x}_2 \end{array}\right]$$

$$x^T C x = \underbrace{x_1^T A x}_{\geq 0} + \underbrace{x_2^T B x_2}_{\geq 0} \geq 0 \quad \blacksquare$$

(e) Hadamard (elementwise) product : $\circ$

$$A, B \succeq 0 \quad \Rightarrow \quad A \circ B \succeq 0$$

Hint: $\quad ab^T \circ cd^T = (a \circ c)(b \circ d)^T$

$$[ab^T \circ cd^T]_{ij} = a_i b_j c_i d_j = a_i c_i b_j d_j = [(a \circ c)(b \circ d)^T]_{ij}$$

e.g. $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}[b_1, b_2] \circ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}[d_1, d_2] = \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \circ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)\left([b_1, b_2] \circ [d_1, d_2]\right)$

Since $A, B \succeq 0$, we know that $A = \sum_i a_i a_i^T$

& $B = \sum_j b_j b_j^T$

$$A \circ B = \sum_i a_i a_i^T \circ \sum_j b_j b_j^T \quad \text{(for some vectors } a_i \text{ \& } b_j\text{)}$$

[property (iii) of PSD]

$$= \sum_{i,j} a_i a_i^T \circ b_j b_j^T$$

$$= \sum_{i,j} (a_i \circ b_j)(a_i \circ b_j)^T$$

$$= \sum_{i,j} c_{ij} c_{ij}^T \quad \text{where } \vec{c}_{ij} = \vec{a}_i \circ \vec{b}_j$$

$$= CC^T$$ where columns of matrix C are the column vectors $\vec{c}_{ij}$

Since $A \circ B = CC^T$ for some matrix C, $A \circ B \succeq 0$ ■

Essence of the proof

$$A = \sum_i a_i a_i^T \quad \& \quad B = \sum_j b_j b_j^T$$

$$A \circ B = \sum_i a_i a_i^T \circ \sum_j b_j b_j^T$$

Since Hadamard operation ($\circ$) distributes over addition ($+$), all we need to prove is $aa^T \circ bb^T \succeq 0 \;\forall a, b$
and use the fact that sum of PSD is PSD

$$(aa^T \circ bb^T)_{ij} = a_i a_j b_i b_j$$

$$x^T (aa^T \circ bb^T) x = \mathrm{Tr}\left((xx^T)^T, aa^T \circ bb^T \right)$$

$$= \mathrm{Tr}\left(xx^T (aa^T \circ bb^T) \right)$$

$$= \langle xx^T, aa^T \circ bb^T \rangle$$

Frobenius inner product of matrices

$$= \mathrm{Sum}\left(xx^T \circ aa^T \circ bb^T \right)$$

$$= \sum_{i,j} x_i x_j a_i a_j b_i b_j$$

$$= \sum_{ij} \underbrace{(x_i a_i b_i)}_{c_i} \underbrace{(x_j a_j b_j)}_{c_j}$$

$$= \sum_{ij} c_i c_j = \left(\sum_k c_k \right)^2 = \left(\sum_k x_k a_k b_k \right)^2 \geq 0$$

■

(f) $A \succ 0 \Rightarrow A^{-1} \succ 0$

positive definite

Proof Let $\lambda, \vec{v}$ be an eigenvector, eigenvalue pair of A

$$A\vec{v} = \lambda \vec{v}$$

$$\Rightarrow A^{-1}A\vec{v} = \lambda A^{-1}\vec{v}$$

$$\Rightarrow \mathbb{I}\vec{v} = \lambda A^{-1}\vec{v}$$

$$\Rightarrow \frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$$

$\Rightarrow \frac{1}{\lambda}, \vec{v}$ is an eigenvalue eigenvector pair for A^{-1}

Since $A \succ 0$, all eigenvalues $\lambda_i > 0 \Rightarrow \frac{1}{\lambda_i} > 0$

$\Downarrow$

all eigenvalues of A^{-1} are +ve

$\Downarrow$

A^{-1} is positive definite

QUICK CHECKS FOR NOT PSD

i) M is not symmetric

ii) $\text{Trace}(M) < 0$

This is because $\text{Tr}(M)$ = Sum of eigenvalues of M

iii) $\text{Det}(M) < 0$

This is because $\text{Det}(M)$ = product of eigenvalues of M.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$\frac{1}{2} x^T (A + A^T) x$$

$$\frac{1}{2} x^T A x + x^T A^T x$$