CSCI 567 Discussion: Linear Algebra Review II

Jan 23, 2026

Linear Combinations and Span

- The **span** of a set of vectors $\{x_1, x_2, \ldots x_n\}$ is the set of all vectors that can be expressed as a linear combination of $\{x_1, \ldots, x_n\}$. That is,

$$\mathrm{span}(\{x_1, \ldots x_n\}) = \left\{ v : v = \sum_{i=1}^{n} \alpha_i x_i, \;\; \alpha_i \in \mathbb{R} \right\}$$

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- A set of vectors $\{x_1, x_2, \ldots x_n\} \subset \mathbb{R}^m$ is said to be **linearly dependent** if one vector belonging to the set can be represented as a linear combination of the remaining vectors, e.g.,

$$x_n = \sum_{i=1}^{n-1} \alpha_i x_i$$

Rank

- **Column rank:** largest number of columns that constitute a linearly independent set.
- **Row rank:** largest number of rows that constitute a linearly independent set.
- Column rank of any matrix is equal to its row rank.
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- Properties ($A \in \mathbb{R}^{m \times n}$):
  - $\text{rank}(A) \leq \min(m, n)$. If $\text{rank}(A) = \min(m, n)$, A is said to be **full rank**.
  - $\text{rank}(A) = \text{rank}(A^T)$.
  - For $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$, $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.
  - For $A, B \in \mathbb{R}^{m \times n}$, $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

Inverse of a Matrix

- A matrix A is **invertible** (or **non-singular**) if there exists a matrix A^{-1} such that

$$A^{-1}A = I_n = AA^{-1}$$

- Interpretation: the transformation A can be "undone"

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- It turns out that:
    - Only square matrices can be invertible
    - When inverses exist, they are unique
    - A is invertible precisely if it has full rank. (Equiv., nonzero determinant!)

Exercise

Prove the following properties, for non-singular $A, B \in \mathbb{R}^{n \times n}$:

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^T = (A^T)^{-1}$, denoted by A^{-T}

Determinant

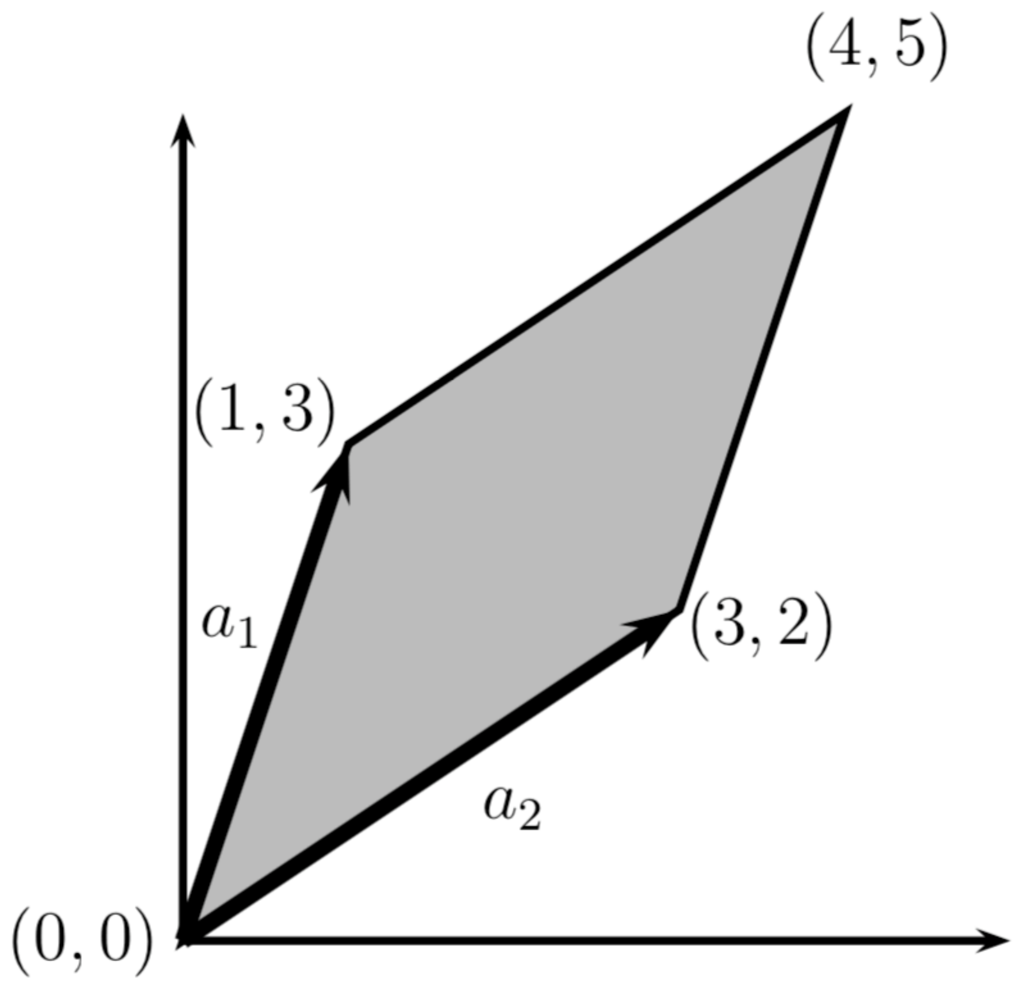
- Let $A \in \mathbb{R}^{n \times n}$, a_i denotes its ith column; consider the set of points $S \subset \mathbb{R}^n$:

$$S = \{v \in \mathbb{R}^n : v = \sum_{i=1}^{n} \alpha_i a_i \ \ (0 \leq \alpha_i \leq 1; \ \ i = 1, \ldots, n)\}$$

- The **determinant** of A, denoted $\det(A)$ or $|A|$, is the 'signed volume' of S

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad a_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



Determinant

(Recursive) Formula

- Let $A \in \mathbb{R}^{n \times n}$, $A_{\backslash i, \backslash j} \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix that results from deleting the ith row and jth column from A

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{\backslash i, \backslash j}| \qquad (\forall\ j \in 1, \ldots, n)$$

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- Equations for small matrices:

$$\Big| [a] \Big| = a \qquad \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

Determinant

- Properties ($A, B \in \mathbb{R}^{n \times n}$):
  - $|A| = |A^T|$
  - $|AB| = |A||B|$
  - $|A| = 0$ iff A is singular
  - For non-singular A, $|A^{-1}| = 1/|A|$

Exercise

- Consider some vector $x \in \mathbb{R}^n$. What is the rank of the matrix xx^T?

Eigenvalues

Key Idea: Sometimes matrices behave simply along special directions.

Are there vectors whose direction does not change under A?

A nonzero vector v is an **eigenvector** of a matrix A if

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Theorem: Every real $n \times n$ matrix has exactly n eigenvalues in $\mathbb{C}$, counting multiplicity (i.e., repetitions)

Symmetric Matrices & Quadratic Forms

Symmetric Matrices: From now, we will assume that matrices A are symmetric, meaning $A = A^\top$.

Symmetric matrices have the following properties:

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A **quadratic form** is a function of the form

$$f(x) = x^\top A x.$$

Appears in least squares, ridge regression, Gaussian likelihoods, etc.

Positive Semidefinite (PSD) Matrices

A symmetric matrix A is **positive semidefinite** (PSD) if

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Intuition: Ax never "pushes against" the vector x.

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A **positive definite** matrix A is symmetric with $x^\top A x > 0 \quad \forall x \neq 0$. (Equivalently, all eigenvalues > 0.)

Exercise

Q1 Which of the following statements are true? PSD stands for positive semi-definite.

(a) $XX^\top$ is a PSD matrix if and only if X is PSD.

(b) If X and Y are PSD matrices, then so is $\lambda X + \mu Y$ for any $\lambda, \mu \in \mathbb{R}$.

(c) If $X - Y$ and $X + Y$ are PSD matrices, then so are X and Y.

(d) All eigenvalues of a symmetric PSD matrix are non-negative.

Exercise

Q2 Suppose A and B are two positive definite matrices. Which matrix may NOT be positive definite?

(a) A^{-1}

(b) $A + B$

(c) $AA^\top$

(d) $A - B$

Exercise

Q3 Consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(a) Compute $x^\top A x$ for $x = (x_1, x_2)^\top$.

(b) Is A positive semidefinite?

(c) Is A positive definite?

(d) Is A invertible?

Exercise

Q4 Suppose A is a PSD matrix and M is any (not necessarily square) matrix of compatible dimensions. Prove that $M^\top AM$ is PSD.

Exercise

Q5 Let A be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$.

(a) What are the eigenvalues of $A + \mu I$ for $\mu \in \mathbb{R}$?

(b) For what values of μ is $A + \mu I$ positive semidefinite?